

COMPUTATION OF EIGENVALUE-EIGENVECTOR AND HARMONIC MOTION SOLUTION FOR LAMINATED RUBBER-METAL SPRING

M.A. Salim^{1,*}, M.R. Mansor², M.T. Musthafah³, M.Z. Akop⁴,
A. Putra⁵ and M.A. Abdullah⁶

^{1,2,3,4,5,6} Faculty of Mechanical Engineering, Universiti Teknikal Malaysia Melaka,
Hang Tuah Jaya, 76100 Durian Tunggal, Melaka, Malaysia.

ABSTRACT

This paper presents the modeling of multi-degree-of-freedom on laminated rubber-metal spring in axial direction displacement. Two methods are used, the first method is the eigenvalues and eigenvectors solution and the second method is harmonic motion solution. In eigenvalues and eigenvectors approach, an equation of motion of laminated rubber-metal spring is developed using a spring-mass system. Then, the equation was rewritten again in matrix and harmonic motion in order to reduce the difficulty and become realistic to be solved using characteristic equation. On the other hand, harmonic motion approach is started from governing equation in term of mode shape. By using this concept, two important equations are finally derived which are displacement and velocity. Using these two methods, finally the maximum displacements of laminated rubber-metal spring are plotted as well as in frequency domain axis. Two types of analysis are considered in this study which are undamped and damped system. Based on the results obtained, the maximum displacement occurred at the undamped system. By increasing the number of degree-of-freedom, the displacement is slowly reduced.

KEYWORDS: *Laminated rubber-metal spring; isolator; eigenvalue; eigenvector*

1.0 INTRODUCTION

Recently, in order to reduce the unwanted vibration from powertrain in automotive engine transferred to the car structure, various mounting systems have been proposed by scientists and engineers (Arib Rejab et al., 2013; Pan, Yang & Yan, 2014). There are several types of these mounting system which are hydraulic mounting, active mounting, semi-active mounting, elastomeric mounting and many more. All of the mountings usually use natural rubber as the principle material. At low frequency, this mounting functions successfully but at high frequency, the response to the unwanted vibration can be questioned. Overall, many researchers have agreed the range of low frequency is in the range of 1 to 80 Hz and for high frequency the range started from 120 Hz. The range is valid for automotive areas including powertrain system (Pan et al., 2014; Shi, Wu, Lloyd & Li, 2014; Xie, Yu & Li, 2013). In order to overcome the problem faced in high frequency range of vibration, new mounting or isolator has been introduced called laminated rubber-metal spring (LR-MS). This new isolator can be categorized into multi-degree-of-freedom (m-DOF). In single-DOF (single-

*Corresponding Email: azli@utem.edu.my

DOF), the isolator was developed using fully natural rubber (NR) in rod molding. In two-DOF (two-DOF), metal plate is added in NR rod and place at the center. Metal plate is proposed to divide NR rod into two sections. When it occurred, the NR rod is called two-DOF of LR-MS. In order to increase the number of DOF, number of the metal plate is increased. All the response in transmissibility effect can be referred from previous publication (Salim, Putra, Thompson, Ahmad & Abdullah, 2013; Salim, Abdullah, Putra, 2014a;2014b;2014c; Abdullah, Salim, Putra, 2014).

In this paper, eigenvalues and eigenvectors solution is used to determine the maximum displacement occurred in m-DOF of LR-MS. Using this solution, the physical behavior of LR-MS was transferred to the equation of motion (EOM). Based on the EOM, the equation was rewritten again into matrix form and harmonic motion. It is a necessary procedure to make it easy and becomes realistic to be solved using characteristic equation. The details are discussed in the next section. Furthermore, this paper also describes on the method of harmonic motion solution to solve LR-MS in axial direction displacement. In this method, it is derived from governing equation in term of mode shape. Finally, two important equations have successfully been derived which are displacement and velocity. However, this study is only focused on axial direction displacement.

2.0 RESEARCH METHODOLOGY

2.1 Eigenvalue and Eigenvector

The full scale schematic diagram for the experimental of LR-MS for single- DOF is shown in Figure 1 (Salim et al., 2014a). According to Figure 1, the LR-MS is coupled with a flexible foundation at the top part and rigid foundation at the bottom part. The flexible foundation represents the free boundary condition while the other side represents the fixed boundary condition. The free boundary condition is necessary to give a free movement to a shaker for excitation force from flexible foundation to LR-MS. On the other hand, fixed boundary condition is applied to block any movement from LR-MS including displacement, rotational and moment.

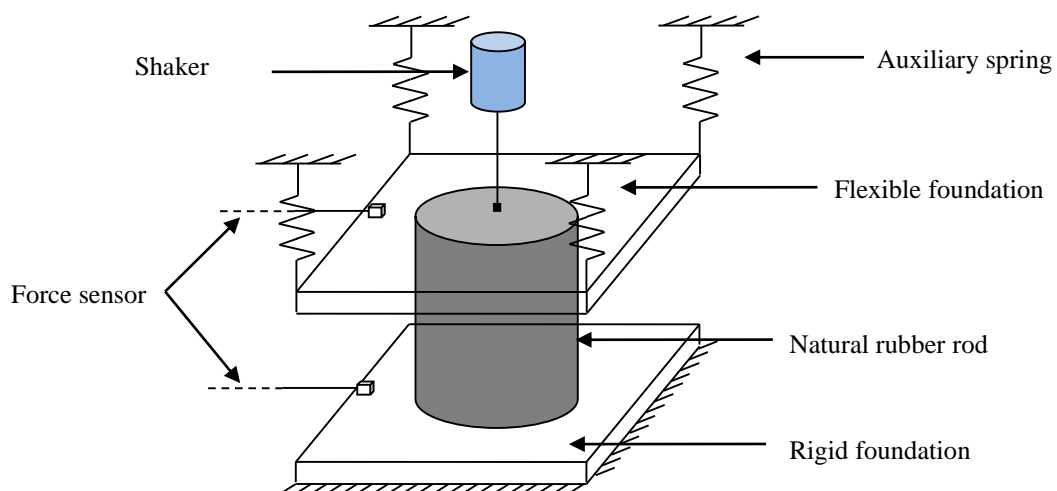


Figure 1. Full scale schematic diagram for the experiment of laminated rubber-metal spring

Figure 2 shows the LR-MS system for single-DOF and it is standalone from the experimental test rig and Figure 3 shown two-DOF of LR-MS. The criteria of single-DOF of LR-MS, the rod is developed using NR material which is known as a compressible behavior which is accurate to represent as a vibration isolator. From single-DOF of LR-MS, two pieces of metal plates are placed at the top and the bottom of LR-MS system. Overall, it is standard procedure to evaluate the strength and compressible behaviors of LR-MS itself before further analysis can be made. Through this procedure, it can be described by a spring-mass system and presented in Figure 4. In this system, m_1 and m_2 represents as a flexible and rigid foundation, respectively. Then, k_1 represents as a stiffness of LR-MS and y_1 , y_2 are displacements in y-direction.

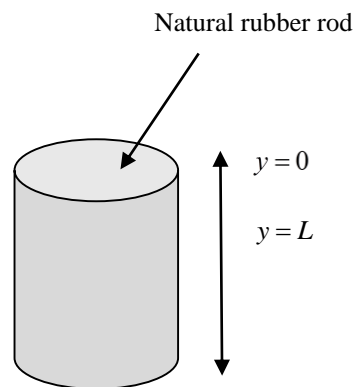


Figure 2. Single-degree-of-freedom for laminated rubber-metal spring

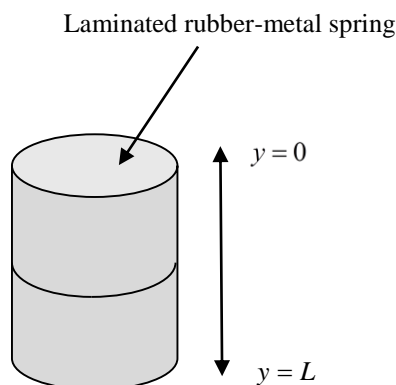


Figure 3. Two-degree-of-freedom for laminated rubber-metal spring

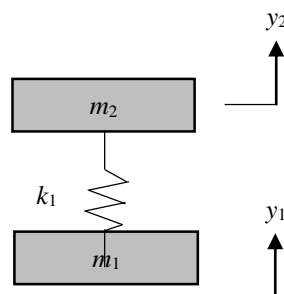


Figure 4. Basic schematic diagram for laminated rubber-metal spring coupled with two solid plate

To ensure the quantum of strength and compressible on the LR-MS system, the method called eigenvalues and eigenvector are used to observe the mode shape behavior for LR-MS system. Basically, the equation of motion is derived from Figure 4 and it is given by

$$m_1 \ddot{y}_1 + k_1 y_1 - k_1 y_2 = 0 \quad (1)$$

and

$$m_2 \ddot{y}_2 + k_1 y_2 - k_1 y_1 = 0 \quad (2)$$

In matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

In harmonic motion:

$$\left(\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4)$$

By expending Equation (4), the new equation can be written as

$$\begin{bmatrix} k_1 - \omega^2 m_1 & -k_1 \\ -k_1 & k_1 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5)$$

The solution is determinant equal to zero and used to make homogenous equation. The equation can be represented as

$$\begin{vmatrix} k_1 - \omega^2 m_1 & -k_1 \\ -k_1 & k_1 - \omega^2 m_2 \end{vmatrix} = 0 \quad (6)$$

For characteristic equation:

$$(k_1 - \omega^2 m_1)(k_1 - \omega^2 m_2) - (-k_1)(-k_1) = 0 \quad (7)$$

By expending Equation (7), the new equation is

$$k_1^2 - k_1 \omega^2 m_2 - k_1 \omega^2 m_1 + \omega^4 m_1 m_2 - k_1^2 = 0 \quad (8)$$

By applying factorial concept, Equation (8) is become

$$\omega^4 m_1 m_2 - k_1 \omega^2 m_1 - k_1 \omega^2 m_2 = 0 \quad (9)$$

Presume that $\omega^2 = X$ and then substitute into Equation (9). The new equation is becoming

$$X^2 m_1 m_2 - X k_1 m_1 - X k_1 m_2 = 0 \quad (10)$$

or

$$X^2 m_1 m_2 - X(m_1 + m_2)k_1 = 0 \quad (11)$$

Quadratic solution is used and two equations are created as

$$X = \frac{(m_2 + m_1)k}{m_2 m_1} \quad (12)$$

and

$$X = 0 \quad (13)$$

The eigenvalues equation then can be written as:

$$\omega^2 = k \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \quad (14)$$

or

$$\omega^2 = 0 \quad (15)$$

For eigenvectors equation, there are two cases which are

$$\begin{bmatrix} k_1 - \omega^2 m_1 & -k_1 \\ -k_1 & k_1 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} \longleftarrow \text{Numerator - Case 1} \\ \longleftarrow \text{Denominator - Case 2} \end{matrix} \quad (16)$$

Using Equation (16), the matrix can be solved and solution (For case 1), is shown below.

$$(k_1 - \omega^2 m_1)y_1 + (-k_1)y_2 = 0 \quad (17)$$

By expanding Equation (17), the equation becomes

$$\left[k_1 - \left[k_1 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) m_1 \right] \right] y_1 - k_1 y_2 = 0 \quad (18)$$

or

$$\left[k_1 - k_1 m_1 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \right] y_1 - k_1 y_2 = 0 \quad (19)$$

By simplifying Equation (19) into less denominator, the equation becomes

$$\left[k_1 - \frac{k_1 m_1}{m_1} - \frac{k_1 m_1}{m_2} \right] y_1 - k_1 y_2 = 0 \quad (20)$$

or

$$\left[k_1 - k_1 - \frac{k_1 m_1}{m_2} \right] y_1 - k_1 y_2 = 0 \quad (21)$$

From Equation (21), the standalone stiffness equation can be removed. The new equation becomes

$$\left[-\frac{k_1 m_1}{m_2} \right] y_1 - k_1 y_2 = 0 \quad (22)$$

By stabilizing for both hand side (HS), Equation (22) can be written as

$$k_1 y_2 = - \left[\frac{k_1 m_1}{m_2} \right] y_1 \quad (23)$$

By transferring negative sign into left hand side (LHS), Equation (23) now becomes

$$- \frac{k_1 y_2}{k_1 y_1} = \frac{m_1}{m_2} \quad (24)$$

Equation (24) can be simplified and finally the case 1 solution is

$$\frac{y_2}{y_1} = - \frac{m_1}{m_2} \quad (25)$$

For case 2, the procedure for the solving the matrix is similar to case 1. Firstly, the equation of motion in Equation (16) can be represented as

$$(-k_1)y_1 + (k_1 - \omega^2 m_2)y_2 = 0 \quad (26)$$

Then, Equation (26) can be expanding to another form to make it easy to solve and it can be written as

$$(-k_1)y_1 + \left[k_1 - \left[k_1 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) m_2 \right] \right] y_2 = 0 \quad (27)$$

or

$$-k_1 y_1 + \left[k_1 - \left[k_1 m_2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \right] \right] y_2 = 0 \quad (28)$$

By simplifying Equation (28) into neglected m_2 as a denominator, the new equation is become

$$-k_1 y_1 + \left[k_1 - \left(\frac{k_1 m_2}{m_1} + \frac{k_1 m_2}{m_2} \right) \right] y_2 = 0 \quad (29)$$

or

$$-k_1 y_1 + \left[k_1 - \frac{k_1 m_2}{m_1} - k_1 \right] y_2 = 0 \quad (30)$$

Standalone stiffness as expressed in Equation (30) can be reduced because they has same quantum but only different in magnitude. Therefore the equation can be written as

$$-k_1 y_1 - \left[\frac{k_1 m_2}{m_1} \right] y_2 = 0 \quad (31)$$

By relocating the position of both displacement, Equation (31) can be represented as

$$k_1 y_1 = - \left[\frac{k_1 m_2}{m_1} \right] y_2 \quad (32)$$

The negative sign now is transferred into LHS, the equation becomes

$$-\frac{k_1 y_1}{k_1 y_2} = \frac{m_2}{m_1} \quad (33)$$

The solution of case 2 can be solved into two solutions which are

$$\frac{y_1}{y_2} = - \frac{m_2}{m_1} \quad (34)$$

or

$$\frac{y_2}{y_1} = -\frac{m_1}{m_2} \quad (35)$$

Equation (25) and Equation (34) can be used in higher order of LR-MS system.

Based on the derived Equation (34) and Equation (35), solutions derived in this study significantly proved that for the LR-MS, the input force is equal to the output force, which correctly satisfies the conservation law. However, if, at a condition where the input force is equal to output force, the proposed LR-MS material is not effective to absorb the load given, thus the laminated spring added to the LR-MS will enable higher input force to be absorbed during operation. With non-addition of the laminator, the current isolator will also fail due to bulging effect at maximum load condition

2.2 Harmonic Motion Solution

Eigenvalues and eigenvectors are the methods to determine the motion in each mode which has a fixed shape. The shape for mode one is represented in Equation (25) and shape for mode two in Equation (35), respectively. In mode shape 1, it must be in time harmonic, which is plotted in frequency domain results, and the shape is proportional with mass ratio. For mode shape 2, the elaboration is still valid as the same as in mode shape 1. Mode shape 1 also can be written in harmonic motion where the governing equation is

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} e^{i\omega_1 t} = \phi^{(1)} M_1 e^{i\omega_1 t} \quad (36)$$

where Y_1 and Y_2 are values of mode shape, $\phi^{(1)}$ shape at 1, $e^{i\omega_1 t}$ is harmonic motion in term of frequency domain and M_1 is modal amplitude at mode shape 1.

Equation (36) also can be rewrite in another form which is

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \phi^{(1)} A_1 \cos \omega_1 t + \phi^{(1)} B_1 \sin \omega_1 t \quad (37)$$

where A_1 and B_1 are modal amplitude and ω_1 is quantum of frequency at mode shape 1.

For mode shape 2, it is also can be written in harmonic motion and they is

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \phi^{(2)} M_2 e^{i\omega_2 t} \quad (38)$$

where $\phi^{(2)}$ shape at 2, and M_2 is modal amplitude at mode shape 2.

Equation (38) can be rewritten again into harmonic motion and the new equation is

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \phi^{(2)} A_2 \cos \omega_2 t + \phi^{(2)} B_2 \sin \omega_2 t \quad (39)$$

where A_2 and B_2 are modal amplitude and ω_2 is quantum of frequency at mode shape 2.

In superposition, it can be done only in free vibration. This requirement is still valid for LR-MS system shown in Figure 3. This phenomenon happen in each of the two modes of vibration as discussed before. Equation (36) and Equation (38) can be merged into one equation and it can be expressed as

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \phi^{(1)} M_1 e^{i\omega_1 t} + \phi^{(2)} M_2 e^{i\omega_2 t} \quad (40)$$

and

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = (\phi^{(1)} A_1 \cos \omega_1 t + \phi^{(1)} B_1 \sin \omega_1 t) + (\phi^{(2)} A_2 \cos \omega_2 t + \phi^{(2)} B_2 \sin \omega_2 t) \quad (41)$$

According to Equation (40) and Equation (41), the actual motion of LR-MS system is depending on the initial condition. Based on the equation, the initial condition can be assumed using the value time variant t or displacement of mass Y . By taking $t=0$ and displacement at mass 1 is unity for both amplitudes, mass 2 is still considered as unity for upper side and -0.5 for bottom side, then both masses are considered released from rest, the general formulation can be stated as

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} A_1 \cos \omega_1 t + \phi^{(1)} B_1 \sin \omega_1 t + \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix} A_2 \cos \omega_2 t + \phi^{(2)} B_2 \sin \omega_2 t \quad (42)$$

Equation (42) can be separated as well into displacement for mass 1 and mass 2. The separated equations are shown below.

$$Y_1(t) = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t + A_2 \cos \omega_2 t + B_2 \sin \omega_2 t \quad (43)$$

and

$$Y_2(t) = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t - 0.5 A_2 \cos \omega_2 t - 0.5 B_2 \sin \omega_2 t \quad (44)$$

Equation (43) and Equation (44) can be written in velocities equation by differentiating the equation by time variant. The new equations become

$$\dot{Y}_1(t) = -\omega_1 A_1 \sin \omega_1 t + \omega_1 B_1 \cos \omega_1 t - \omega_2 A_2 \sin \omega_2 t + \omega_2 B_2 \cos \omega_2 t \quad (45)$$

and

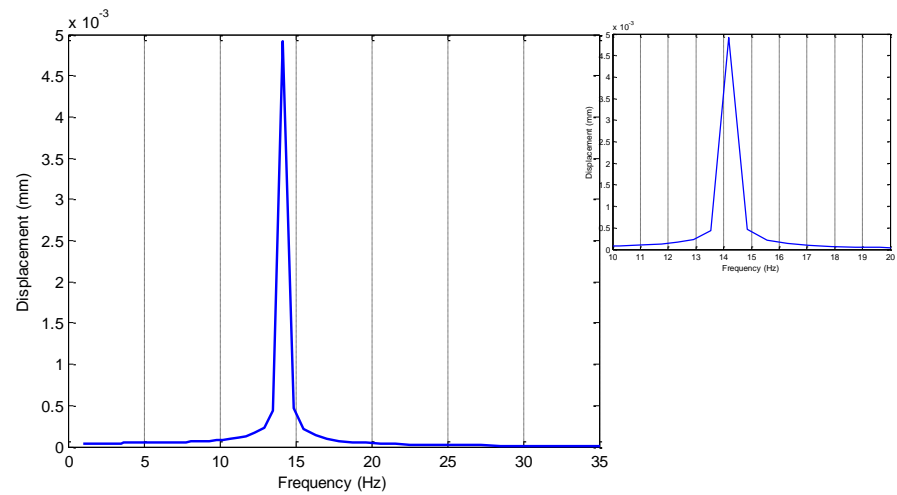
$$\dot{Y}_2(t) = -\omega_1 A_1 \cos \omega_1 t + \omega_1 B_1 \sin \omega_1 t + \omega_2 0.5 A_2 \cos \omega_2 t - \omega_2 0.5 B_2 \sin \omega_2 t \quad (46)$$

3.0 RESULTS AND DISCUSSION

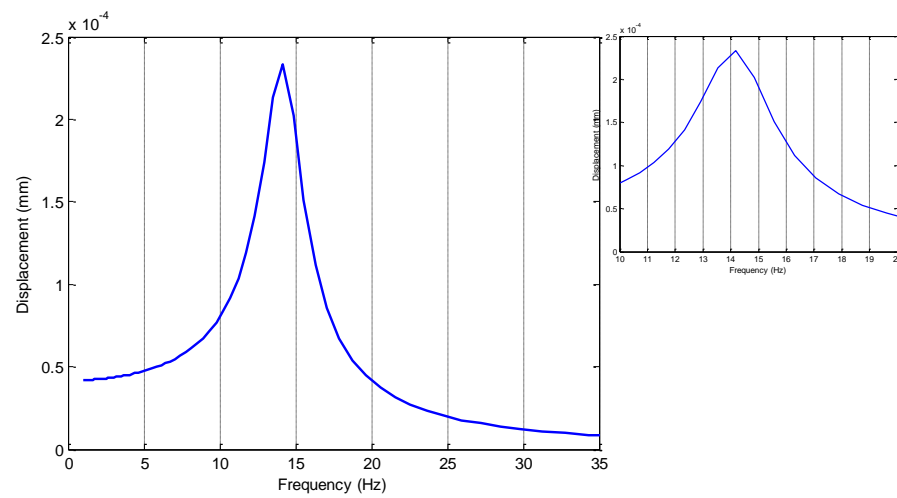
Equation (43) and Equation (44) present the equation of displacement and Equation (45) and Equation (46) present the equation of velocity for LR-MS system. These equations can be used for both cases either in undamped and damped system for LR-MS. In this study, it focused on the displacement. All of the results are shown in Figure 5 until Figure 7. In Figure 5 (a) and (b), the graph shows the result for undamped and damped system for single-DOF of LR-MS. The maximum peak for displacement locates at the same place in the range of frequency of 14 Hz but totally different in the value of displacement. At undamped system, the maximum displacement is recorded at 5×10^{-3} mm and for damped system is 2.5×10^{-4} mm. According to the displacement data, the displacement for undamped system is higher than damped system. For the assumption, it happens because in undamped system, the damping coefficient is not playing the role and because of this reason, the peak is high without any damping energy influencing the phenomenon.

In two-DOF of LRMS, there are two peaks recorded at 9 and 23 Hz. These two peaks represented the first and second natural frequency of the system respectively. The locations of these two natural frequencies are located at the same place for both cases either in undamped and damped systems. The straight line represents bottom mass and dotted line is for top mass. The quantum of displacement for both masses are totally different where the bottom mass, it is recorded at 2.0×10^{-3} mm and 4.2×10^{-4} mm for both undamped and damped systems. For the top mass, the maximum displacements are recorded at 3.4×10^{-3} mm for undamped system and 7.0×10^{-4} mm for damped system. The displacement patterns are quite the same compared with the single-DOF of LR-MS system.

In order to ensure the maximum displacement affected by damping coefficient, three-DOF of LR-MS system should be developed. With this system, it is recorded to have three peaks at 6, 17 and 26 Hz. All three of these peaks represents the natural frequency of the system. In this system, there are three lines for both graphs which represent the bottom, center and top masses. Overall, the displacement is totally different although locates at the same frequency. For the undamped system, the maximum displacement are recorded at 1.5, 3.0 and 4.0×10^{-3} mm for bottom, center and top masses. In the damped system, the maximum displacement are recorded at 0.6, 1.0 and 1.4×10^{-3} mm. The maximum displacement recorded at three-DOF of LR-MS system. It can be concluded that the maximum displacement of LR-MS system is influenced by damping coefficient. By increasing the value of damping coefficient, it can reduce the displacement of the system.

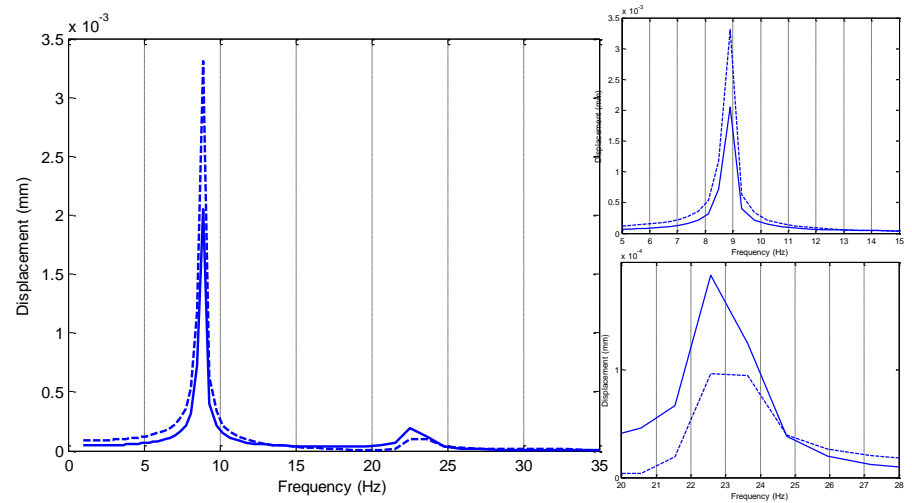


(a)

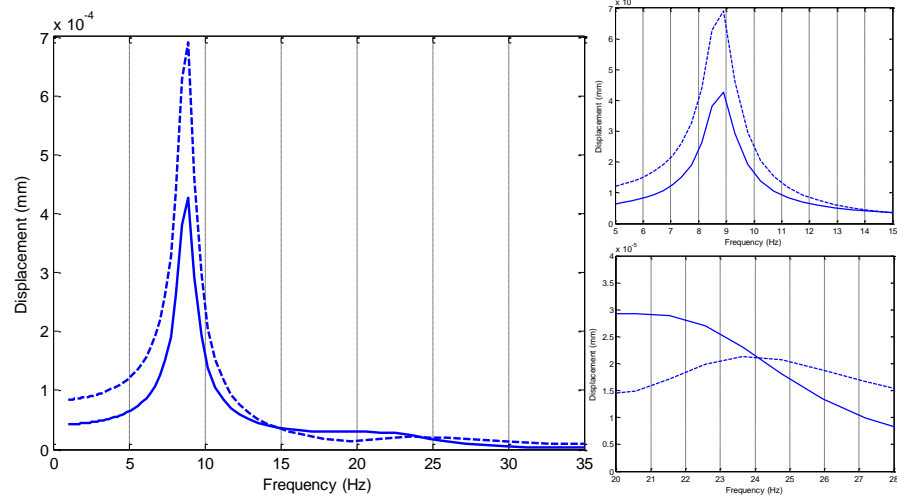


(b)

Figure 5. Single-degree-of-freedom: (a) undamped and (b) damped system



(a)



(b)

Figure 6. Two-degree-of-freedom: (a) undamped and (b) damped system
(—) 1st displacement and (--) 2nd displacement

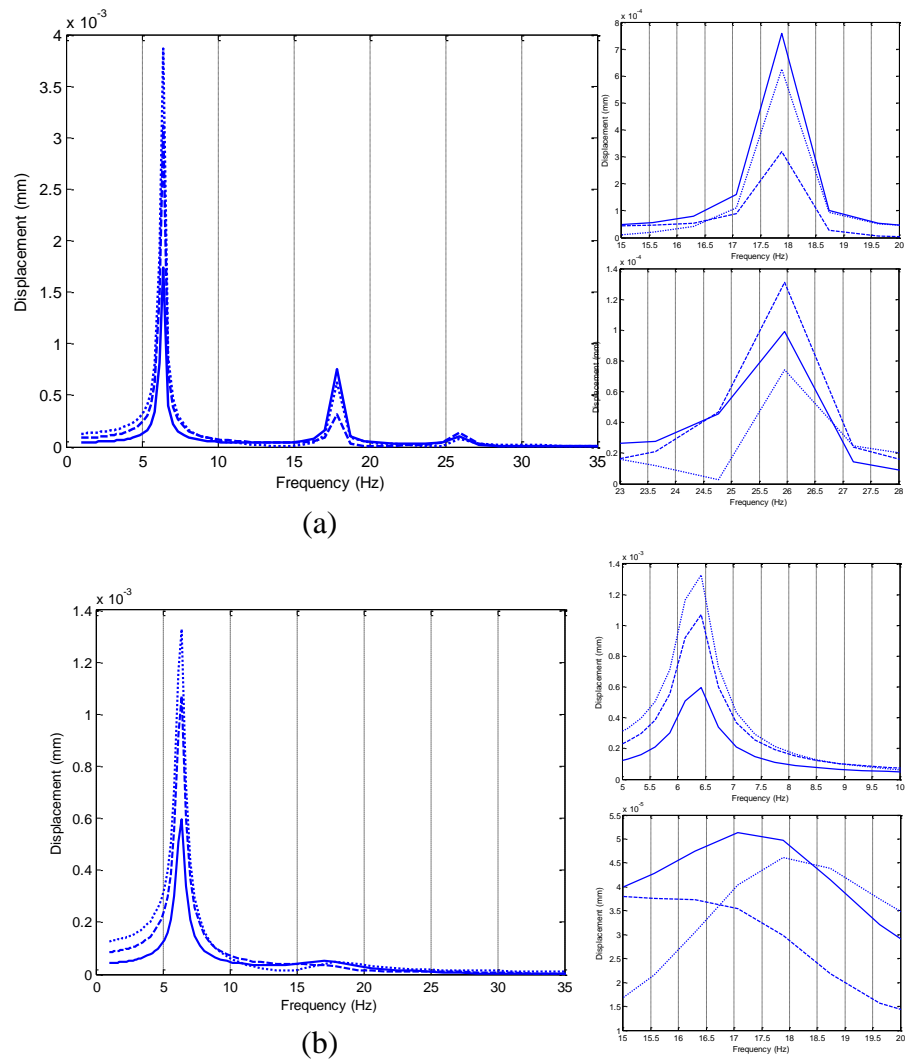


Figure 7. Three-degree-of-freedom: (a) undamped and (b) damped system
(—) 1st displacement, (--) 2nd displacement and (···) 3rd displacement

In addition, the displacement of each mass is depending separately for a system with more than one mass. This has been proven successful where two-DOF and three-DOF of LR-MS systems, the maximum displacement occurs separately for each mass. Therefore, Equation (25) and Equation (26) are valid for m-DOF of LR-MS system. Then, Equation (43) and Equation (44) is still valid in the method of determining the maximum displacement, but modal amplitude should be provided to produce the solution.

4.0 CONCLUSION

Maximum displacement in the axial direction for LR-MS model was studied in this paper. Two methods have been used, the first method is eigenvalues and eigenvectors whereas the second method is harmonic motion solution. By using these two methods, the maximum displacement are plotted into frequency domain. Two types of analyses are used which are the undamped and the damped system of laminated rubber-metal spring. For single-DOF, the maximum displacement is occurred at undamped system compare other one. The analysis

is continued to two-DOF and three-DOF. The results have shown the undamped system represented maximum displacement compared to damped system.

ACKNOWLEDGEMENT

This project is funded by Universiti Teknikal Malaysia Melaka under the Short Term Research Grant Scheme PJP/2012/FKM (42A)/S01046, PJP/2013/FKM (5B)/S01164, and the Ministry of Education (MoE), Malaysia under Fundamental Research Grant Scheme (FRGS), FRGS/2013/FKM/TK06/02/2/F00165. This project also a collaboration among Green Technology Vehicles Research Group (GTev), Innovative Machine and Mechanism Research Group (INNOMA) and Advanced Vehicle Technology Research Group (AcTiVe) at Faculty of Mechanical Engineering, Universiti Teknikal Malaysia Melaka.

REFERENCES

- Abdullah, M.A., Salim, M.A., Putra, A. (2014). Linear and angular displacement relationship of natural rubber engine isolator. *Applied Mechanics and Material*, 575: 250-253.
- Arib Rejab, M.N., Abd Rahman, R., Raja Hamzah, R.I, Inayat Hussain, J.I., Ahmad, N., Ismail, A., Putra, A. (2013). Fabricating process and mechanical properties of elastomeric mount. The 7th South East Asia Technical University Consortium (SEATUC) Symposium, 97-102.
- Pan, G.Y., Yang, X., Yan, Y. (2014). Optimization of vehicle powertrain mounting system and its performance. *Applied Mechanics and Materials*, 580-583.
- Salim, M.A., Abdullah, M.A., Putra, A. (2014a). Predicted transmissibility of an experimental approach for a laminated rubber-metal spring. *American-Eurasian Journal of Sustainable Agriculture*, 8: 104-110.
- Salim, M.A., Putra, A., Abdullah, M.A. (2014b). Analysis of axial vibration in the laminated rubber-metal spring. *Advanced Materials Research*, 845: 46-50.
- Salim, M.A., Putra, A., Abdullah, M.A. (2014c). Mathematical formulation for bending moment wave in non-dispersive finite rod. *International Review on Modelling and Simulations*, 7:523-530.
- Salim, M.A., Putra, A., Thompson, D.J., Ahmad, N., Abdullah, M.A. (2013). Transmissibility of a laminated rubber-metal spring: A preliminary study. *Applied Mechanics and Materials*, 393: 661-665.
- Shi, Y., Wu, P.D., Lloyd, D.J., Li, D.Y. (2014). Effect of rate sensitivity on necking behavior of a laminated tube under dynamic loading. *Journal of Applied Mechanics, Transactions ASME*, 81(5):051010-051010-10.
- Xie, Z., Yu, D., Li, R. (2013). Multi-objective robust optimization design of vehicle engine mount systems. *QicheGongcheng/Automotive Engineering*, 35(10): 893-897.